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CENTRE DE RECHERCHE SUR LES TRANSPORTS
UNIVERSITÉ DE MONTRÉAL
C.P. 6128, SUCCURSALE A.
MONTRÉAL
CANADA H3C 3J7

TÉLÉPHONE: (514) 343-7575
TÉLEX: 05-24146 BIBPOLYTEC
EMAIL: crt@crt.umontreal.ca
TÉLÉCOPIEUR (FAX): (514) 343-7121

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Vehicle Routing with Stochastic Demands and Restricted Failures

by
Moshe Dror[1,2]
Gilbert Laporte[2]
François V. Louveaux[3]

[1] Decision Sciences Group, College of Business and Public Administration, University of Arizona, Tucson, Arizona 85721, U.S.A.
[2] Centre de recherche sur les transports, Université de Montréal, C.P. 6128, Succursale A, Montréal, Canada H3C 3J7.
[3] Facultés Universitaires Notre-Dame de la Paix, 8 Rempart de la Vierge, B-5000 Namur, Belgium.

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Abstract

In this paper we examine the vehicle routing problem (VRP) with stochastic customer demands from the perspective of solution methodologies originally developed for the deterministic routing environment. In the case where the problem is formulated as a chance constrained model a direct implementation of deterministic VRP algorithm is described. For the stochastic programming with recourse approach we examine a number of problem scenarios with limited number of potential route failures. In each case, the solution procedure is based on solving a number of Traveling Salesman Problems (TSP). Different problems and recourse policies require a different number of corresponding TSP solutions. Similarly, the more restrictive the assumption on potential route failures the fewer TSPs need to be solved.

Keywords: stochastic vehicle routing, recourse, traveling salesman problem.

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Résumé

Dans cet article, on étudie une classe de problèmes de tournées de véhicules (PTV) pour lesquels la demande des clients est stochastique. L'approche utilisée consiste à appliquer à ce type de problème des méthodes de résolution originellement conçues pour le cas déterministe. Dans le cas de problèmes avec contraintes probabilistes, on obtient une solution directement par une modification simple d'algorithmes conçus pour le PTV déterministe. Dans le cas de problèmes avec recours, on considère un certain nombre de scénarios avec nombre limité de pannes. La procédure consiste alors à résoudre une série de problèmes du voyageur de commerce (PVC). Le nombre de PVC résolus dépend alors du scénario considéré : il est plus élevé dans le cas de politiques de recours générales, et davantage restreint dans le cas de politiques plus restrictives.

Mots-clés: problèmes stochastiques de construction de tournées de véhicules, recours, problème du voyageur de commerce.

Introduction

The deterministic *Vehicle Routing Problem* (VRP) can be defined as follows. Let $G = (V, A)$ be a graph where $V = \{0, 1, \ldots, n\}$ is a set of vertices representing cities or customers, with vertex 0 denoting a depot, and $A = \{(i, j) : i, j \in V, i \neq j\}$ is an arc set. Also define on A a non-negative cost or distance matrix $C = (c_{ij})$. With every vertex $i > 0$ is associated a known non-negative demand ξ_i to be delivered by a vehicle. A fleet of m identical vehicles of capacity Q is based at the depot. The problem consists of designing a set of m vehicle routes whose cost is minimized in such a way that (i) every route starts and ends at the depot; (ii) every remaining city is visited exactly once by exactly one vehicle; (iii) the sum of demands on any vehicle route does not exceed Q. The VRP and several of its variants have been extensively studied in the Operations Research literature. For recent surveys see Laporte and Nobert (1987) and Golden and Assad (1988).

In the *Stochastic Vehicle Routing Problem* (SVRP), demands ξ_i are random variables. As a result, planned routes must account for the possibility that vehicle capacity my be exceeded by the actual demands at some point on the route. In such case, a *failure* is said to occur and a *recourse action* must be taken. The recourse action may take different forms, some restricted such as returning to the depot to reload and resume deliveries along the planned route, or visiting the remaining customers in a different order in reoptimizing sequence. Clearly the recourse action may include *breaks* at some points along the route which can be planned a priori (restricted recourse) or decided upon as the actual demands are revealed (Dror and Trudeau, 1986; Dror, Laporte, and Trudeau, 1989; Laporte and Louveaux, 1990a). In addition, the routing sequence may be constructed dynamically each time actual customer demand is revealed (Dror,

1990).

The SVRP has been traditionally studied within the framework of Stochastic Programming. It is common to distinguish between *Chance Constrained Programming* (CCP) and *Stochastic Programming with Recourse* (SPR). The CCP formulations of the SVRP usually consist of minimizing the sum of planned route costs, while ensuring that the probability of route failure does not exceed a given threshold α. Stewart and Golden (1983) and Laporte, Louveaux, and Mercure (1989) have shown that under mild assumptions, chance constrained SVRPs reduce to deterministic VRPs. Their solution does not, however, provide information on the cost of recourse action in case of route failure. The SPR version of the SVRP is to minimize both the expected costs of the planned routes plus the costs of recourse actions, subject to (i) of the deterministic case. This problem is considerably more difficult to solve than the chance constrained SVRP. A general Integer Programming Formulation of the SVRP with recourse is provided in Dror (1990) and in Laporte and Louveaux (1990a). In a different paper, Laporte and Louveaux (1990b) have indicated how this problem can be solved to optimality by an *Integer L-Shaped Method*, although there is no guarantee this procedure will work for large problem sizes. A third class of problems, known as the *Bounded Penalty* SVRP has also been studied by Laporte, Louveaux, and Mercure (1989). Here, the objective is to minimize total planned route costs, subject to the constraint that the expected recourse cost of any vehicle route does not exceed a preset fraction of its planned cost. The solution to this problem can be used to provide an upper bound on the cost of the optimal SVRP solution (Laporte and Louveaux, 1990a).

In several practical contexts, particulary when the coefficient of variation of the ξ_i's is small, the number of failures on a planned route is limited and it may be advantageous to exploit this information to construct SVRP solutions

with known bounds on the value of the optimal solution. In some cases, it may even be possible to determine the optimal solution or, failing this, to compute the probability of a heuristic solution being optimal. Moreover, in this context of limited number of failures, it is possible to exploit the relatively vast accumulated knowledge in solving deterministic VRPs for constructing solutions to SVRPs. The object of this paper is to examine such cases. In Section 1, it is assumed that there always exists a set of vehicle routes having at most k failures each; in Section 2, we consider the particular case where $m = 1$ and failures can occur once, at one of the last h customers of the route. In Section 3, we briefly discuss the use of parallel computing for this class of problems. The conclusions follow in Section 4.

1. Problems with a maximum of k failures

We assume that the random variables ξ_i are such that $P(\xi_i > Q) = 0$ for all i and that $P(\sum_{i \in T} \xi_i > a)$ can be efficiently computed for any real number a and for any non-empty subset T of V. This is the case for several common distributions (the Poisson, Normal, and Gamma distributions, for example). Define VRP(k) as the problem consisting of designing m least total cost vehicle routes, subject to constraints (i), (ii) of the VRP and (iv): the probability of having more than k failures on any route is zero. Note that VRP(k) represents a restricted version of the SVRP. Let z_k be the optimal solution value of VRP(k). The problem can be simply solved as a standard VRP in which routes with more than k potential failures are eliminated. Let x_{ij} be a binary variable equal to 1 if and only if arc(i, j) appears in the optimal solution. The formulation of VRP(k) is then

$$minimize \sum_{i \neq j} c_{ij} x_{ij} \qquad (1)$$

subject to

$$\sum_{i=1}^{n} x_{i0} = \sum_{j=1}^{n} x_{0j} = m \tag{2}$$

$$\sum_{i \neq l} x_{il} = \sum_{j \neq l} x_{lj} = 1 \quad (l \in V \setminus \{0\}) \tag{3}$$

$$\sum_{i,j \in T} x_{ij} \leq |T| - v(T,k) \quad (T \subset V \setminus \{0\}, 2 \leq |T| \leq n-1) \tag{4}$$

$$x_{ij} \in \{0,1\} \quad ((i,j) \in A) \tag{5}$$

This formulation of the SVRP belongs to the category of chance constrained models. In this formulation, constraints (2), (3), and (5) are self-explanatory. In subtour elimination constraints (4), $v(T,k)$ is the minimum number of vehicles required so that the probability of there being more than k failures in T is negligeable, i.e. $v(T,k)$ is the smallest integer satisfying $P(\sum_{i \in T} \xi_i > Qv(T,k)) \cong 0$. Clearly, no single route can have more than k failures. These constraints are direct extension of similar constraints used in the VRP (Laporte, Nobert, and Desrochers, 1985) and in another chance constrained SVRP (Laporte, Louveaux, and Mercure, 1989).

It is simple to extend this model to require no more than h ($\geq k$) failures on all the routes combined in addition to limiting the number of failures on any single route to at most k. The formulation is identical to (1) - (5) but for the value m in constraint (2) which is set to satisfy $n - m \leq n - v(V \setminus \{0\}, h)$ since for m routes $\sum_{i,j \in V} x_{ij} \equiv n + m$. Denote by $z_{k,h}$ the optimal solution value of this extended problem with at most h failures overall. (Note that k can be set to 1 in which case each route can have at most one route failure and h failures overall.)

If C is asymmetric, the problem (1) - (5) can be solved quite simply as in Laporte, Mercure, and Nobert (1986), by simple adaptation of the Carpaneto

and Toth (1980) algorithm, for example:

Step 1: Transform the unconstrained VRP into a *Traveling Salesman Problem* (TSP) by using the graph extension proposed by Lenstra and Rinnooy Kan (1975).

Step 2: Apply to the TSP the Carpaneto and Toth algorithm, by considering as illegal not only subtours disconnected from $\{0\}$, but also all paths corresponding to vehicle routes violating constraints (4).

Step 3: Interpret the TSP on the extended graph as a VRP solution on the original graph. □

If C is symmetric, it may be more efficient to resort to Integer Linear Programming. For example, one can use a polytopal cutting plane algorithm (Padberg and Rinaldi, 1990) where, in addition to subtour elimination constraints (4), cuts such as 2-matching inequalities, comb inequalities, and clique tree inequalities are generated.

When the total number of failures allowed h is large (say $h > m$) and the values of demands ξ_i are bounded below by $\xi_i' > 0$ then a lower bound on value of optimal SVRP solution with at most h failures (denoted by z_h^s) stronger than $z_{k,h}$ can be computed similarly as in Laporte and Louveaux (1990a). Denote by $v = max\{\lceil \sum_{i \in V} \xi_i'/Q \rceil - m; 0\}$, the minimum number of route failures in any feasible solution. Let f be the sum of v smallest values of $(c_{i0} + c_{0i}), i \in V$. Clearly $z_{k,h} + f \leq z_h^s$. Another, stronger bound can be computed as follows: Compute, $p_j = P(\sum_{i \in V} \xi_i > (m + j - 1)Q), j = 1, \ldots, km$ as lower bounds on $P($ number of failure $\geq j)$. Then, let f be the sum of km smallest $(c_{i0} + c_{0i}, i \in V$ each multiplied by the appropriate p_j. Again, $z_{k,h} + f \leq z_h^s$. (Note that beyond certain value of $k, p_j = 0$ limiting the number of usefull p_j computations.)

In addition, an upper bound on z_h^s can be establish as follows. Let p_r^s be the probability that the r^{th} route of the VRP(h) solution fails s times ($s = 1, \ldots, h$). For $s \geq 1$, let R^s be the set of s vertices i of that route having the largest values $c_{0i} + c_{i0}$. Then the expected cost of recourse action on route r is no more than $F_r = \sum_{s=1}^{h} p_r^s \sum_{i \in R^s} (c_{i0} + c_{0i})$ and therefore, $z_h^s \leq z_{k,h} + \sum_{r=1}^{m} F_r$.

2 Problems with a maximum of 1 failure on a route

A number of interesting results can be derived in the case where the ξ_i's are such that failure can occur only once, at one of the last k vertices of a route. We first restrict our results to $m = 1$. For this basic case we examine a number of different recourse policies; (i) assume that customers are visited sequentially on the planned route until a failure occurs (if a failure does not occur the route is executed to completion as planned). In such a case, the vehicle returns to the depot to reload and resumes its deliveries as planned from the vertex where failure occured, (ii) cutomers are visited sequentially on the planned route until a failure occurs at which point the vehicle returns to the depot to reload and optimizes its remaining deliveries, and (iii) the order of the customers' deliveries are determined sequentially with each new demand which becomes known upon the vehicle arrival at the vertex.

Recourse Policy (i): For $k > 1$, let $V_k = \{i_1, \ldots, i_k, 0\}$ be the k^{th} ordered subset of cardinality $k + 1$ of V including vertex 0; if $k = 1$ define V_k as $\{0\}$. Denote by i^* the first element of V_k (if $k > 1, i^* = i_1$; if $k = 1, i^* = 0$). Let z_k^1 be the cost of an optimal SVRP solution (under this recourse policy) in which a failure occurs (if at all) at one of the V_k customer vertices. The value of z_k^1 is computed by adding together (a) the cost of an optimal Hamiltonian path starting at 0 and ending at i^*, containing all vertices in $V \backslash V_k$; (b) $\sum_{t=1}^{k-1} c_{i_t, i_{t+1}} +$

$c_{i_k 0}$; (c) the expected cost of the recourse in V_k; i.e. $\sum_{t=1}^{k} p_t(c_{i_t 0} + c_{0 i_t})$, where p_t is the probability of failure at vertex i_t $(p_t = P(A_V + \sum_{r=1}^{i_t - 1} \xi_{i_r} \leq Q < A_V + \sum_{r=1}^{i_t} \xi_{i_r}$, where $A_V = \sum_{i \in V \setminus V_k} \xi_i)$.

Clearly, one would like to select the 'best' subset V_k which requires the resolution of up to $\binom{n}{k}$ (deterministic) TSPs. Thus, finding such optimal SVRP solution implies solving $O(n^k)$ TSPs. We will return to this point latter.

Recourse Policy (ii): Similarly to the previous case, assume that if a failure occurs on a route, it can happen only once at one of the vertices of $V_k \setminus \{0\}$. Let z_k^2 be the cost of an optimal SVRP solution under this recourse policy. The computation of the z_k^2 is more involved than for policy (i). It requires some added notation. Denote by z_W the value of the optimal TSP solution over set $W \subseteq V, \{0\} \in W$. Clearly in computing the value of z_k^2 the first term (a) is the same as for (i). Since we will reoptimize the remaining deliveries in case of a route failure, in order to compute the terms (b) and (c) denote by $p_{i_{\varphi[j]}}$ the probability of route failure occuring exactly at vertex $i_{\varphi[j]}$ given that the vertices in V_k initially are planned to receive their deliveries according to the sequence described by a noncyclic permutation φ of elements in V_k with $\varphi[k+1] = 0$ (the depot). Now we can compute the combined terms (b) and (c) (denoted bellow as (b,c)) as follows:

$$(\text{b,c}) = p_{i^*}(c_{i^*,0} + z_{V_k}) + p_{\varphi[2]}(c_{i^*,\varphi[2]} + c_{\varphi[2],0} + z_{V_k \setminus \{i^*\}}) + p_{\varphi[3]}(c_{\varphi[2],\varphi[3]} + c_{\varphi[3],0} + z_{V_k \setminus \{i^*,\varphi[2]\}}) + \ldots, + p_{\varphi[k]}(c_{\varphi[k-1],\varphi[k]} + 2c_{\varphi[k],0} + c_{0,\varphi[k]}) + (1 - \prod_{t=1}^{k} p_{\varphi[t]}) \sum_{t=1}^{k} c_{\varphi[t],\varphi[t+1]}$$

where $i^* = \varphi[1]$.

In order to select the 'best' V_k subset, the permutation φ, and compute the value z_{V_k}, we would have to solve optimally $O(kn^k)$ TSPs.

Recourse Policy (iii): This is the most general recourse policy (unrestricted) in terms of its recourse action reacting to each new demand information. In this section we do not present a solution outline or discuss the algorithmic complexity for the general case of recourse policy (iii). In the next section we examine the special case of (iii) for a subset $V_k = V_1$.

Finally, suppose that the condition for V_k (i.e., $P(\sum_{i \in V \setminus V_k} \xi_i > Q) = 0$ for all subsets V_k of V is not necessarily satisfied. Then, the probability that z_k^1 and z_k^2 do not correspond to their respective optimal values is equal to at most $1 - max_{V_k} P(\sum_{i \in V \setminus V_k} \xi_i > Q)$. To illustrate, let $n = 10, Q = 100$ and $\xi_i \simeq N(10, 4)$ for all i. Then,

$$P(\sum_{i \in V \setminus V_k} \xi_i) = \begin{cases} 0.0475 & \text{if } |V_k| = 1 \\ 0.0002 & \text{if } |V_k| = 2 \\ 0.0000 & \text{if } |V_k| = 3 \end{cases}$$

In other words, solving a single TSP yields a solution where the probability of being optimal exceeds 95 percent; when $k \geq 2$, it is almost certain that the solution will be optimal. In a sense, the value k in V_k can be selected taking into account the probability of not obtaining the optimal SVRP solution in a given problem setting. Increasing k and solving either recourse policy version (i) or (ii) would clearly increase the probability of obtaining an optimal solution in those cases. The subsequent computational burden would then be weighted relatively to the probability of not obtaining an optimal solution.

3 Maximum of 1 failure on a route at the last customer

Suppose the demand distribution is such that the probability of failure is zero on any subset of $n - 1$ customers (i.e. $V_k = V_1$ and if a failure occurs, this can happen only at the last customer of the tour). Clearly, the probability of a

route failure in this case is a constant, say $B > 0$. For this special case we no longer need to solve n TSPs, one for each customer as the last customer on the tour, but only one TSP on an expanded graph and make an additional simple comparison test. This section focuses on the appropriate graph transformation.

Construct from the original graph $G = (V, A)$ the following directed graph $G' = (V', A')$:

$V' = \{0_1, 0_2, 0_3, i_1, i_2\}, i = 1, \ldots, n$

$A' = \{(0_3, 0_2), (0_1, i_1), (i_1, i_2), (i_2, 0_1), (i_2, j_1), (0_2, i_2), (i_2, 0_1)\}, i = 1, \ldots, n; j = 1, \ldots, n; i \neq j$

The cost matrix $C' = (c'_{ij})$ is defined as follows:

$c'_{0_3,0_2} = 0 = c'_{i_1,i_2}, i = 1, \ldots, n$

$c'_{0_1,i_1} = c_{0i}, i = 1, \ldots, n$

$c'_{i_1,0_3} = Bc_{i0}, c'_{0_2,i_2} = Bc_{0i}, i = 1, \ldots, n$

$c'_{i_2,0_1} = c_{i0}, i = 1, \ldots, n$

$c'_{i_2,j_1} = c_{ij}, i, j = 1, \ldots, n; i \neq j$

We illustrate the graph transformation in Figure 1.

Figure 1 about here

Given the graph in Figure 1, if a solution constructs a delivery sequence $(0, 1, 2, 3, 0)$ (customer 3 as the last customer on the route), the expected value of this solution corresponds to the following sequence of vertices on graph G': $(0_1, 1_1, 1_2, 2_1, 2_2, 3_1, 0_3, 0_2, 3_2, 0_1)$. Similarly, any Hamiltonian circuit over G' which 'breaks' at the last customer on the sequence, corresponds to the expected value of a particular single route solution. An optimal TSP solution on G' is

equivalent to solving n times a TSP problem over G each time with a prespecified last customer node. This transformation of G into G' is possible because of the fixed probability of route failure (at the last customer on the route).

In case the optimal TSP solution for G' 'breaks' at some node other than the last customer on the tour, an additional simple test is required to finalize the optimal customer sequence. Assume that the TSP solution for G' 'breaks' at node i_1 which is not the last customer. Suppose the TSP tour goes like: $(\ldots i_1, 0_3, 0_2, i_2, j_1 \ldots)$ and say customer k is the last one on the tour. Denote by $\Delta = (c_{i0} + c_{oj}) - B(c_{k0} + c_{0k})$ (the first term reflects the fact that after breaking in the middle of the route we would not return to the same customer i but visit the next one j). The value of the Δ expresses the distance 'saved' by breaking early. If $\Delta \leq 0$ then the optimal TSP sequence would have a 'break' at some middle point on the tour. If $\Delta > 0$, then in the optimal sequence the 'break' is reassigned to the last customer on the tour (i.e. k instead of i).

4 A note on parallel computing

Since even in the simple case of one potential route failure per route, we need in general to solve a number $(O(n^k))$ of deterministic TSPs, a possible solution strategy is the use of parallel computing technology (see Kindervater and Lenstra,1988, and Pruul, Nemhauser and Rushmeier, 1988). Our problem seems to best be suited for the general class of parallel computation of the asynchronous class of MIMD (multiple instruction stream, multiple data stream). The processors in this class run independently and wait only if instructions from the other processors are required. Parallel solution inplementation for a single TSP in the MIMD context were attempted by Pruul (1975) who parallelized a deapth first branch-and-bound process. Trienekens (1986)

also describes a parallel branch-and-bound algorithm for the TSP based on the master-slave principle, where the master keeps tracks of all nodes considered for branching, updates the values and receives new values from the slaves. A more flexible MIMD environment is described in Finkel and Manber (1987). Here, a collection of processors, each with its own storage capacity, are connected and cooperate by exchanging messages. Finkel and Manber describe a DIB (Distributed Implementation of Backtracking) package for the division and distribution of computations which is "efficient, completely dynamic and quite general". It was implemented on a Crystal multicomputer at the University of Wisconsin-Madison. Problems solved using this approach included TSPs.

In our case, several TSPs must be solved in parallel. A possible DIB oriented approach would start processing a number of TSPs on different processors, with reallocation of computing power and termination of individual TSP computations each time a new upper bound is determined. Depending on whether the problems are symmetric or not, branch-and-bound codes as those of Padberg and Rinaldi (1990), Carpaneto and Toth (1980), or Balas and Christofides (1981), could be used for the (partial) solution of the individual TSPs.

5. Conclusion

The SVRP with recourse is notoriously difficult to solve to optimality. However, it is often realistic to assume that the potential number of failures per route is limited. For a given problem instance, one usually can preprocess customer demand information to determine the likelihood of such events. Known VRP (deterministic) algorithms can then be adopted to obtain a good feasible SVRP solution and an upper bound on its departure from optimality. When it is known that only one failure can occur at one of the last k vertices of a

route, an optimal solution for number of recourse policies can be derived by solving $O(n^k)$ TSPs. In the current state of knowledge, this is quite reasonable provided n and k are relatively small. For larger parameter values. a possibility would be to use parallel computing. When the size of the problem precludes exact solutions, a small value of k can be selected and one can compute the probability of the resulting solution being optimal. In this sense, we control the probability of obtaining an optimal solution. In the special case when it is known that the probability of failure over any subset of $n-1$ customers is zero, an optimal solution can be computed by solving only one TSP.

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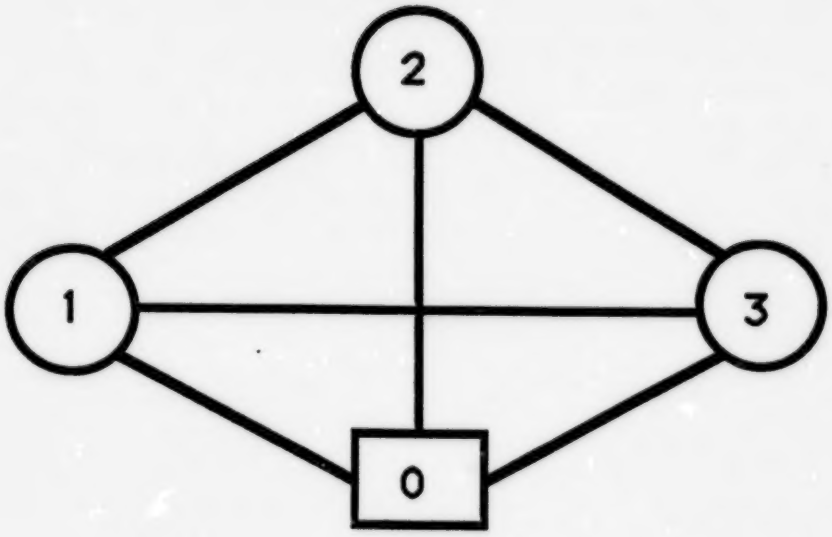
G =



G' =